

# Generalizations of Buzano inequality for $n$ -tuples of vectors in inner product spaces with applications

S. S. Dragomir<sup>1,2</sup>

<sup>1</sup>Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia

<sup>2</sup>DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

E-mail: sever.dragomir@vu.edu.au

## Abstract

In this paper some generalizations of Buzano inequality for  $n$ -tuples of vectors in inner product spaces are given. Applications for norm and numerical radius inequalities for  $n$ -tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

2010 Mathematics Subject Classification. **47A05**. 47A63, 47A99

Keywords. Inner product spaces, Schwarz inequality, Buzano inequality, Numerical radius, Operator norm, Functions of normal operators.

## 1 Introduction

Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex numbers field  $\mathbb{K}$ . The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\| \|y\| \geq |\langle x, y \rangle| \text{ for any } x, y \in H. \quad (1.1)$$

The equality case holds in (1.1) if and only if there exists a constant  $\lambda \in \mathbb{K}$  such that  $x = \lambda y$ .

In 1985 the author [5] (see also [18]) established the following refinement of (1.1):

$$\|x\| \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| \quad (1.2)$$

for any  $x, y, e \in H$  with  $\|e\| = 1$ .

Using the triangle inequality for modulus we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \geq |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|$$

and by (1.2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \\ &\geq 2|\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|, \end{aligned}$$

which implies the Buzano inequality [3]

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y \rangle|] \geq |\langle x, e \rangle \langle e, y \rangle| \quad (1.3)$$

that holds for any  $x, y, e \in H$  with  $\|e\| = 1$ .

For other Schwarz related inequalities in inner product spaces, see [1], [6]-[10], [16], [17], [21], [22], [23], [24], [25], [26], [27], [28] and the monographs [13] and [14].

Let  $(H; \langle \cdot, \cdot \rangle)$  be a complex Hilbert space. The *numerical range* of an operator  $T$  is the subset of the complex numbers  $\mathbb{C}$  given by [20, p. 1]:

$$W(T) = \{\langle Tx, x \rangle, x \in H, \|x\| = 1\}. \quad (1.4)$$

It is well known (see for instance [20]) that:

- (i) The numerical range of an operator is convex (the Toeplitz-Hausdorff theorem);
- (ii) The spectrum of an operator is contained in the closure of its numerical range;
- (iii)  $T$  is self-adjoint if and only if  $W$  is real.

The *numerical radius*  $w(T)$  of an operator  $T$  on  $H$  is defined by [20, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x \rangle|, \|x\| = 1\}. \quad (1.5)$$

It is well known that  $w(\cdot)$  is a norm on the Banach algebra  $B(H)$  and the following inequality holds true

$$w(T) \leq \|T\| \leq 2w(T), \text{ for any } T \in B(H). \quad (1.6)$$

Utilising Buzano's inequality (1.3) we obtained the following inequality for the numerical radius [11] or [12]:

**Theorem 1.1.** Let  $(H; \langle \cdot, \cdot \rangle)$  be a Hilbert space and  $T : H \rightarrow H$  a bounded linear operator on  $H$ . Then

$$w^2(T) \leq \frac{1}{2} [w(T^2) + \|T\|^2]. \quad (1.7)$$

The constant  $\frac{1}{2}$  is best possible in (1.7).

From the above result (1.7) we obviously have

$$w(T) \leq \left\{ \frac{1}{2} [w(T^2) + \|T\|^2] \right\}^{1/2} \leq \left\{ \frac{1}{2} (\|T^2\| + \|T\|^2) \right\}^{1/2} \leq \|T\| \quad (1.8)$$

and

$$w(T) \leq \left\{ \frac{1}{2} [w(T^2) + \|T\|^2] \right\}^{1/2} \leq \left\{ \frac{1}{2} (w^2(T) + \|T\|^2) \right\}^{1/2} \leq \|T\|, \quad (1.9)$$

that provide refinements for the first inequality in (1.6).

For numerical radius inequalities see the recent monograph [15] and the references therein.

Motivated by the above results, we establish some generalizations of Buzano inequality for  $n$ -tuples of vectors in inner product spaces. Applications for norm and numerical radius inequalities for  $n$ -tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

## 2 Main results

We have the following generalization of Buzano's inequality:

**Theorem 2.1.** Let  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$  be a probability distribution, i.e. we recall that  $p_i > 0$  for any  $i \in \{1, \dots, n\}$  and  $\sum_{i=1}^n p_i = 1$ . For any  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in H^n$  we have

$$\begin{aligned} & \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, e \right\rangle \left\langle e, \sum_{i=1}^n p_i y_i \right\rangle \right| + \left| \left\langle \sum_{i=1}^n p_i x_i, e \right\rangle \left\langle e, \sum_{i=1}^n p_i y_i \right\rangle \right| \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} & \frac{1}{2} \left[ \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} + \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \right] \\ & \geq \left| \left\langle \sum_{i=1}^n p_i x_i, e \right\rangle \left\langle e, \sum_{i=1}^n p_i y_i \right\rangle \right|, \end{aligned} \quad (2.2)$$

for any  $e \in H$  with  $\|e\| = 1$ .

*Proof.* For a probability distribution  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$ , we define the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_p := \sum_{i=1}^n p_i \langle x_i, y_i \rangle$$

for  $n$ -tuples  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in H^n$ .

The attached norm is given by

$$\|\mathbf{x}\|_p = \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2}$$

for  $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ .

Let  $\mathbf{e} = (e_1, \dots, e_n) \in H^n$  with  $\sum_{i=1}^n p_i \|e_i\|^2 = 1$ . Making use of (1.2) and (1.3) for the inner product  $\langle \cdot, \cdot \rangle_p$  we have the inequalities

$$\begin{aligned} & \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \sum_{i=1}^n p_i \langle x_i, e_i \rangle \sum_{i=1}^n p_i \langle e_i, y_i \rangle \right| + \left| \sum_{i=1}^n p_i \langle x_i, e_i \rangle \sum_{i=1}^n p_i \langle e_i, y_i \rangle \right| \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} & \frac{1}{2} \left[ \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} + \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \right] \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, e_i \rangle \sum_{i=1}^n p_i \langle e_i, y_i \rangle \right|, \end{aligned} \quad (2.4)$$

for any  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in H^n$ .

If we take  $\mathbf{e} = (e, \dots, e) \in H^n$  with  $\|e\| = 1$ , then we get from (2.3) and (2.4) the desired inequalities (2.1) and (2.2). ■

**Remark 2.2.** If we take in (2.1) and (2.2)  $n = 1$  and  $p_1 = 1$ , then we get the inequalities (1.2) and (1.3).

We observe that, if we take  $H = \mathbb{C}$  with the inner product  $\langle z, w \rangle = z\bar{w}$  then by taking above  $x_i = a_i \in \mathbb{C}, y_i = \bar{b}_i, i \in \{1, \dots, n\}$  and  $e = 1$  then from (2.1) and (2.2) we get the inequalities

$$\begin{aligned} & \left( \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i |b_i|^2 \right)^{1/2} \\ & \geq \left| \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \right| + \left| \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \right| \\ & \geq \left| \sum_{i=1}^n p_i a_i b_i \right| \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} & \frac{1}{2} \left[ \left( \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i |b_i|^2 \right)^{1/2} + \left| \sum_{i=1}^n p_i a_i b_i \right| \right] \\ & \geq \left| \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \right|. \end{aligned} \quad (2.6)$$

We have the following norm inequality:

**Theorem 2.3.** Let  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$  be a probability distribution and  $(A_1, \dots, A_n), (B_1, \dots, B_n)$  two  $n$ -tuples of bounded linear operators on  $H$ . Then we have

$$\begin{aligned} & \frac{1}{2} \left[ \left\| \sum_{i=1}^n p_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n p_i |B_i|^2 \right\|^{1/2} + \left\| \sum_{i=1}^n p_i B_i^* A_i \right\| \right] \\ & \geq \left\| \sum_{i=1}^n p_i A_i e \right\| \left\| \sum_{i=1}^n p_i B_i e \right\| \geq \left| \left\langle \left( \sum_{i=1}^n p_i B_i^* \right) \sum_{i=1}^n p_i A_i e, e \right\rangle \right| \end{aligned} \quad (2.7)$$

or any  $e \in H$  with  $\|e\| = 1$ .

*Proof.* If we write the inequality (2.2) for  $x_i = A_i x$ ,  $y_i = B_i y$ , then we get

$$\begin{aligned} & \frac{1}{2} \left[ \left( \sum_{i=1}^n p_i \|A_i x\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|B_i y\|^2 \right)^{1/2} + \left| \sum_{i=1}^n p_i \langle A_i x, B_i y \rangle \right| \right] \\ & \geq \left| \left\langle \sum_{i=1}^n p_i A_i x, e \right\rangle \left\langle e, \sum_{i=1}^n p_i B_i y \right\rangle \right|, \end{aligned} \tag{2.8}$$

for any  $x, y, e \in H$  with  $\|e\| = 1$ .

Observe that

$$\begin{aligned} \sum_{i=1}^n p_i \|A_i x\|^2 &= \sum_{i=1}^n p_i \langle A_i x, A_i x \rangle = \sum_{i=1}^n p_i \langle A_i x, A_i x \rangle \\ &= \sum_{i=1}^n p_i \langle A_i^* A_i x, x \rangle = \sum_{i=1}^n p_i \langle |A_i|^2 x, x \rangle \\ &= \left\langle \sum_{i=1}^n p_i |A_i|^2 x, x \right\rangle, \end{aligned}$$

$$\sum_{i=1}^n p_i \|B_i y\|^2 = \left\langle \sum_{i=1}^n p_i |B_i|^2 y, y \right\rangle$$

and

$$\sum_{i=1}^n p_i \langle A_i x, B_i y \rangle = \left\langle \sum_{i=1}^n p_i B_i^* A_i x, y \right\rangle$$

for any  $x, y \in H$ .

Then by (2.8) we get the inequality

$$\begin{aligned} & \frac{1}{2} \left[ \left\langle \sum_{i=1}^n p_i |A_i|^2 x, x \right\rangle^{1/2} \left\langle \sum_{i=1}^n p_i |B_i|^2 y, y \right\rangle^{1/2} + \left| \left\langle \sum_{i=1}^n p_i B_i^* A_i x, y \right\rangle \right| \right] \\ & \geq \left| \left\langle \sum_{i=1}^n p_i A_i e, x \right\rangle \left\langle \sum_{i=1}^n p_i B_i e, y \right\rangle \right|, \end{aligned} \tag{2.9}$$

for any  $x, y, e \in H$  with  $\|e\| = 1$ , which is an inequality of interest in itself.

Taking the supremum over  $\|x\| = \|y\| = 1$  in (2.9) we get

$$\begin{aligned}
& \sup_{\|x\|=1} \left| \left\langle \sum_{i=1}^n p_i A_i e, x \right\rangle \right| \sup_{\|y\|=1} \left| \left\langle \sum_{i=1}^n p_i B_i e, y \right\rangle \right| \tag{2.10} \\
&= \sup_{\|x\|=\|y\|=1} \left| \left\langle \sum_{i=1}^n p_i A_i e, x \right\rangle \left\langle \sum_{i=1}^n p_i B_i e, y \right\rangle \right| \\
&\leq \frac{1}{2} \sup_{\|x\|=\|y\|=1} \left[ \left\langle \sum_{i=1}^n p_i |A_i|^2 x, x \right\rangle^{1/2} \left\langle \sum_{i=1}^n p_i |B_i|^2 y, y \right\rangle^{1/2} \right. \\
&\quad \left. + \left| \left\langle \sum_{i=1}^n p_i B_i^* A_i x, y \right\rangle \right| \right] \\
&\leq \frac{1}{2} \left[ \sup_{\|x\|=1} \left\langle \sum_{i=1}^n p_i |A_i|^2 x, x \right\rangle^{1/2} \sup_{\|y\|=1} \left\langle \sum_{i=1}^n p_i |B_i|^2 y, y \right\rangle^{1/2} \right. \\
&\quad \left. + \sup_{\|x\|=\|y\|=1} \left| \left\langle \sum_{i=1}^n p_i B_i^* A_i x, y \right\rangle \right| \right]
\end{aligned}$$

for any  $e \in H$  with  $\|e\| = 1$ .

Since

$$\begin{aligned}
& \sup_{\|x\|=1} \left| \left\langle \sum_{i=1}^n p_i A_i e, x \right\rangle \right| = \left\| \sum_{i=1}^n p_i A_i e \right\|, \\
& \sup_{\|y\|=1} \left| \left\langle \sum_{i=1}^n p_i B_i e, y \right\rangle \right| = \left\| \sum_{i=1}^n p_i B_i e \right\|, \\
& \sup_{\|x\|=1} \left\langle \sum_{i=1}^n p_i |A_i|^2 x, x \right\rangle^{1/2} = \left\| \sum_{i=1}^n p_i |A_i|^2 \right\|^{1/2}, \\
& \sup_{\|y\|=1} \left\langle \sum_{i=1}^n p_i |B_i|^2 y, y \right\rangle^{1/2} = \left\| \sum_{i=1}^n p_i |B_i|^2 \right\|^{1/2}
\end{aligned}$$

and

$$\sup_{\|x\|=\|y\|=1} \left| \left\langle \sum_{i=1}^n p_i B_i^* A_i x, y \right\rangle \right| = \left\| \sum_{i=1}^n p_i B_i^* A_i \right\|.$$

Making use of (2.10) we get the first inequality in (2.7).

Using Schwarz inequality in  $(H; \langle \cdot, \cdot \rangle)$  we have

$$\begin{aligned}
& \left\| \sum_{i=1}^n p_i A_i e \right\| \left\| \sum_{i=1}^n p_i B_i e \right\| \geq \left| \left\langle \sum_{i=1}^n p_i A_i e, \sum_{i=1}^n p_i B_i e \right\rangle \right| \\
&= \left| \left\langle \left( \sum_{i=1}^n p_i B_i^* \right) \sum_{i=1}^n p_i A_i e, e \right\rangle \right|
\end{aligned}$$

for any  $e \in H$  with  $\|e\| = 1$ , and the second inequality in (2.7) is proved. ■

**Corollary 2.4.** Let  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$  be a probability distribution and  $(A_1, \dots, A_n)$  an  $n$ -tuple of bounded linear operators on  $H$ . Then we have

$$\begin{aligned} & \frac{1}{2} \left[ \left\| \sum_{i=1}^n p_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n p_i |A_i^*|^2 \right\|^{1/2} + \left\| \sum_{i=1}^n p_i A_i^2 \right\| \right] \\ & \geq \left\| \sum_{i=1}^n p_i A_i e \right\| \left\| \sum_{i=1}^n p_i A_i^* e \right\| \geq \left\langle \left( \sum_{i=1}^n p_i A_i \right)^2 e, e \right\rangle \end{aligned} \quad (2.11)$$

for any  $e \in H$  with  $\|e\| = 1$ .

It follows from (2.7) by taking  $B_i = A_i^*$ ,  $i \in \{1, \dots, n\}$ .

**Remark 2.5.** Taking the supremum over  $\|e\| = 1$  in (2.7) and (2.11), then we get the numerical radius inequalities

$$\begin{aligned} & w \left( \sum_{i=1}^n p_i B_i^* \sum_{i=1}^n p_i A_i \right) \\ & \leq \sup_{\|e\|=1} \left\| \sum_{i=1}^n p_i A_i e \right\| \left\| \sum_{i=1}^n p_i B_i e \right\| \\ & \leq \frac{1}{2} \left[ \left\| \sum_{i=1}^n p_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n p_i |B_i|^2 \right\|^{1/2} + \left\| \sum_{i=1}^n p_i B_i^* A_i \right\| \right] \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} & w \left( \left( \sum_{i=1}^n p_i A_i \right)^2 \right) \\ & \leq \sup_{\|e\|=1} \left( \left\| \sum_{i=1}^n p_i A_i e \right\| \left\| \sum_{i=1}^n p_i A_i^* e \right\| \right) \\ & \leq \frac{1}{2} \left[ \left\| \sum_{i=1}^n p_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n p_i |A_i^*|^2 \right\|^{1/2} + \left\| \sum_{i=1}^n p_i A_i^2 \right\| \right], \end{aligned} \quad (2.13)$$

where  $(A_1, \dots, A_n)$ ,  $(B_1, \dots, B_n)$  are two  $n$ -tuples of bounded linear operators on  $H$ .

We recall that a bounded linear operator  $T$  is *normal* if  $TT^* = T^*T$ . This is equivalent to the fact that  $\|Tx\| = \|T^*x\|$  for any  $x \in H$ .

If  $(A_1, \dots, A_n)$  is an  $n$ -tuple of normal operators on  $H$ , then from the first inequality in (2.11) we get

$$\frac{1}{2} \left[ \left\| \sum_{i=1}^n p_i |A_i|^2 \right\| + \left\| \sum_{i=1}^n p_i A_i^2 \right\| \right] \geq \left\| \sum_{i=1}^n p_i A_i e \right\|^2 \quad (2.14)$$

for any  $e \in H$  with  $\|e\| = 1$ .

Taking the supremum over  $\|e\| = 1$  in (2.14) we also have

$$\frac{1}{2} \left[ \left\| \sum_{i=1}^n p_i |A_i|^2 \right\| + \left\| \sum_{i=1}^n p_i A_i^2 \right\| \right] \geq \left\| \sum_{i=1}^n p_i A_i \right\|^2, \quad (2.15)$$

where  $(A_1, \dots, A_n)$  is an  $n$ -tuple of normal operators on  $H$ .

If we take  $p_i = \frac{q_i}{\sum_{k=1}^n q_k}$  with  $q_i \geq 0, i \in \{1, \dots, n\}$  and  $\sum_{k=1}^n q_k > 0$ , then we get from (2.15)

$$\frac{1}{2} \left[ \left\| \sum_{i=1}^n q_i |A_i|^2 \right\| + \left\| \sum_{i=1}^n q_i A_i^2 \right\| \right] \sum_{k=1}^n q_k \geq \left\| \sum_{i=1}^n q_i A_i \right\|^2, \quad (2.16)$$

for any  $(A_1, \dots, A_n)$  an  $n$ -tuple of normal operators on  $H$ .

If we take  $q_i = r_i^2, A_i = \frac{1}{r_i} T_i$ , where  $r_i$  are nonzero real numbers,  $i \in \{1, \dots, n\}$  and  $(T_1, \dots, T_n)$  is an  $n$ -tuple of normal operators on  $H$ , then from (2.16) we get

$$\frac{1}{2} \left[ \left\| \sum_{i=1}^n |T_i|^2 \right\| + \left\| \sum_{i=1}^n T_i^2 \right\| \right] \sum_{k=1}^n r_k^2 \geq \left\| \sum_{i=1}^n r_i T_i \right\|^2, \quad (2.17)$$

which for  $T_i = z_i 1_H, i \in \{1, \dots, n\}$  produces the *de Bruijn inequality*

$$\frac{1}{2} \left[ \sum_{i=1}^n |z_i|^2 + \sum_{i=1}^n z_i^2 \right] \sum_{k=1}^n r_k^2 \geq \left| \sum_{i=1}^n r_i z_i \right|^2. \quad (2.18)$$

We have the following numerical radius inequality:

**Theorem 2.6.** Let  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$  be a probability distribution and  $(A_1, \dots, A_n)$   $n$ -tuples of bounded linear operators on  $H$ . Then we have

$$\begin{aligned} & w^2 \left( \sum_{i=1}^n p_i A_i \right) \\ & \leq \frac{1}{2} \left[ \left\| \sum_{i=1}^n p_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n p_i |A_i^*|^2 \right\|^{1/2} + w \left( \sum_{i=1}^n p_i A_i^2 \right) \right]. \end{aligned} \quad (2.19)$$

*Proof.* If in (2.9) we take  $B_i = A_i^*, i \in \{1, \dots, n\}$  and  $x = y = e$ , then we get the inequality

$$\begin{aligned} & \frac{1}{2} \left[ \left\langle \sum_{i=1}^n p_i |A_i|^2 e, e \right\rangle^{1/2} \left\langle \sum_{i=1}^n p_i |A_i^*|^2 e, e \right\rangle^{1/2} + \left\langle \sum_{i=1}^n p_i A_i^2 e, e \right\rangle \right] \\ & \geq \left| \left\langle \sum_{i=1}^n p_i A_i e, e \right\rangle \left\langle \sum_{i=1}^n p_i A_i^* e, e \right\rangle \right| = \left| \left\langle \sum_{i=1}^n p_i A_i e, e \right\rangle \right|^2, \end{aligned} \quad (2.20)$$



for any  $e \in H$  with  $\|e\| = 1$ .

By taking the supremum over  $\|e\| = 1$  in (2.20), we get

$$\begin{aligned}
& w^2 \left( \sum_{i=1}^n p_i A_i \right) \\
&= \sup_{\|e\|=1} \left| \left\langle \sum_{i=1}^n p_i A_i e, e \right\rangle \right|^2 \\
&\leq \frac{1}{2} \sup_{\|e\|=1} \left[ \left\langle \sum_{i=1}^n p_i |A_i|^2 e, e \right\rangle^{1/2} \left\langle \sum_{i=1}^n p_i |A_i^*|^2 e, e \right\rangle^{1/2} + \left| \left\langle \sum_{i=1}^n p_i A_i^2 e, e \right\rangle \right| \right] \\
&\leq \frac{1}{2} \left[ \sup_{\|e\|=1} \left( \left\langle \sum_{i=1}^n p_i |A_i|^2 e, e \right\rangle^{1/2} \left\langle \sum_{i=1}^n p_i |A_i^*|^2 e, e \right\rangle^{1/2} \right) \right. \\
&\quad \left. + \sup_{\|e\|=1} \left| \left\langle \sum_{i=1}^n p_i A_i^2 e, e \right\rangle \right| \right] \\
&\leq \frac{1}{2} \left[ \sup_{\|e\|=1} \left( \left\langle \sum_{i=1}^n p_i |A_i|^2 e, e \right\rangle^{1/2} \sup_{\|e\|=1} \left\langle \sum_{i=1}^n p_i |A_i^*|^2 e, e \right\rangle^{1/2} \right) \right. \\
&\quad \left. + \sup_{\|e\|=1} \left| \left\langle \sum_{i=1}^n p_i A_i^2 e, e \right\rangle \right| \right] \\
&= \frac{1}{2} \left[ \left\| \sum_{i=1}^n p_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n p_i |A_i^*|^2 \right\|^{1/2} + w \left( \sum_{i=1}^n p_i A_i^2 \right) \right]
\end{aligned}$$

and the inequality (2.19) is proved. ■

**Remark 2.7.** If we take in (2.19)  $n = 1$  and  $A_1 = T$ , then we recapture the inequality (1.7), namely

$$w^2(T) \leq \frac{1}{2} [w(T^2) + \|T\|^2].$$

The case  $n = 2$  is important since it allows to apply the above inequalities for the Cartesian decomposition of an operator.

If we take in (2.19)  $n = 2$  and  $q_1 = q_2 = \frac{1}{2}$ , then we have

$$w^2(A_1 + A_2) \leq \left\| |A_1|^2 + |A_2|^2 \right\|^{1/2} \left\| |A_1^*|^2 + |A_2^*|^2 \right\|^{1/2} + w(A_1^2 + A_2^2). \quad (2.21)$$

Assume that  $T$  is a bounded linear operator and consider the Cartesian decomposition

$$T = A + iB,$$

with the selfadjoint operators  $A, B$  given by

$$A = \frac{1}{2}(T^* + T), \quad B = \frac{1}{2i}(T - T^*).$$

Take  $A_1 = A$ ,  $A_2 = iB$ . Then  $A_1 + A_2 = T$ ,

$$|A_1|^2 + |A_2|^2 = A^2 + B^2 = \frac{1}{2} (|T|^2 + |T^*|^2),$$

$$|A_1^*|^2 + |A_2^*|^2 = A^2 + B^2 = \frac{1}{2} (|T|^2 + |T^*|^2)$$

and

$$A_1^2 + A_2^2 = \frac{1}{4} (T^* + T)^2 + \frac{1}{4} (T - T^*)^2 = \frac{1}{2} (T^2 + (T^*)^2).$$

Using (2.21) we get

$$w^2(T) \leq \frac{1}{2} \left[ \left( \| |T|^2 + |T^*|^2 \| + w(T^2 + (T^*)^2) \right) \right] \quad (2.22)$$

for any  $T$  a bounded linear operator.

### 3 Applications for functions of normal operators

Recall some examples of power series with nonnegative coefficients

$$\begin{aligned} \frac{1}{1-\lambda} &= \sum_{n=0}^{\infty} \lambda^n, \quad \lambda \in D(0, 1); \\ \ln \frac{1}{1-\lambda} &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n, \quad \lambda \in D(0, 1); \\ \exp(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n, \quad \lambda \in \mathbb{C}; \\ \sinh \lambda &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1}, \quad \lambda \in \mathbb{C}; \\ \cosh \lambda &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n}, \quad \lambda \in \mathbb{C}. \end{aligned} \quad (3.1)$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\begin{aligned} \frac{1}{2} \ln \left( \frac{1+\lambda}{1-\lambda} \right) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0, 1); \\ \sin^{-1}(\lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} (2n+1) n!} \lambda^{2n+1}, \quad \lambda \in D(0, 1); \\ \tanh^{-1}(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0, 1); \\ {}_2F_1(\alpha, \beta, \gamma, \lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha) \Gamma(n+\beta) \Gamma(\gamma)}{n! \Gamma(\alpha) \Gamma(\beta) \Gamma(n+\gamma)} \lambda^n, \quad \alpha, \beta, \gamma > 0, \\ &\lambda \in D(0, 1) \end{aligned} \quad (3.2)$$

where  $\Gamma$  is *Gamma function*.

The following inequality for power series with nonnegative coefficients holds:

**Theorem 3.1.** Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a power series with nonnegative coefficients  $a_n \geq 0$  for  $n \in \mathbb{N}$  and having the radius of convergence  $R > 0$  or  $R = \infty$ . If  $U, V$  are normal operators on the Hilbert space  $(H; \langle \cdot, \cdot \rangle)$ ,  $U^*V = VU^*$  and  $\alpha > 0$  such that  $\alpha < R$  and  $\|U\|, \|V\| \leq 1$ , then

$$\begin{aligned} & |\langle f(\alpha V) e, x \rangle \langle f(\alpha U) e, y \rangle| \\ & \leq \frac{1}{2} \left[ \langle f(\alpha |V|^2) x, x \rangle \langle f(\alpha |U|^2) y, y \rangle + |\langle f(U^*V) x, y \rangle| \right] f(\alpha) \end{aligned} \tag{3.3}$$

for any  $x, y, e \in H$  with  $\|e\| = 1$ .

*Proof.* Using the inequality (2.9) we have for  $n \geq 1$

$$\begin{aligned} & \frac{1}{2} \left[ \left\langle \sum_{i=0}^n a_i \alpha^i |V|^2 x, x \right\rangle^{1/2} \left\langle \sum_{i=0}^n a_i \alpha^i |U|^2 y, y \right\rangle^{1/2} \right. \\ & \left. + \left| \left\langle \sum_{i=0}^n a_i \alpha^i (U^*)^i V^i x, y \right\rangle \right| \right] \sum_{i=0}^n a_i \alpha^i, \\ & \geq \left| \left\langle \sum_{i=0}^n a_i \alpha^i V^i e, x \right\rangle \left\langle \sum_{i=0}^n a_i \alpha^i U^i e, y \right\rangle \right| \end{aligned} \tag{3.4}$$

for any  $x, y, e \in H$  with  $\|e\| = 1$ .

Since  $U, V$  are normal operators, then for  $i \geq 1$

$$|V^i|^2 = (V^i)^* V^i = (V^*)^i V^i = (V^*V)^i = |V|^{2i}$$

and

$$|U^i|^2 = |U|^{2i}.$$

Also, since  $U^*V = VU^*$ , then

$$(U^*)^i V^i = (U^*V)^i$$

for any  $i \geq 1$ .

Then from (3.4) we have

$$\begin{aligned} & \frac{1}{2} \left[ \left\langle \sum_{i=0}^n a_i \alpha^i |V|^{2i} x, x \right\rangle^{1/2} \left\langle \sum_{i=0}^n a_i \alpha^i |U|^{2i} y, y \right\rangle^{1/2} \right. \\ & \left. + \left| \left\langle \sum_{i=0}^n a_i \alpha^i (U^*V)^i x, y \right\rangle \right| \right] \sum_{i=0}^n a_i \alpha^i \\ & \geq \left| \left\langle \sum_{i=0}^n a_i \alpha^i V^i e, x \right\rangle \left\langle \sum_{i=0}^n a_i \alpha^i U^i e, y \right\rangle \right|, \end{aligned} \tag{3.5}$$

for any  $x, y, e \in H$  with  $\|e\| = 1$ .

Since  $\left\| \alpha |V|^2 \right\| = \alpha \|V\|^2 < R$ ,  $\left\| \alpha |U|^2 \right\| = \alpha \|U\|^2 < R$ ,  $\|\alpha U^*V\| \leq \alpha \|U\| \|V\| < R$ ,  $\|\alpha U\| < R$  and  $\|\alpha V\| < R$ , then the series

$$\sum_{i=0}^{\infty} a_i \alpha^i U^i, \quad \sum_{i=0}^{\infty} a_i \alpha^i V^i, \quad \sum_{i=0}^n a_i \alpha^i |V|^{2i}, \quad \sum_{i=0}^n a_i \alpha^i |U|^{2i}, \quad \sum_{i=0}^{\infty} a_i \alpha^i (U^*V)^i$$

are convergent in  $B(H)$  and  $\sum_{i=0}^{\infty} a_i \alpha^i$  is convergent in  $\mathbb{R}$ .

Taking the limit over  $n \rightarrow \infty$  in (3.5) we get the desired result (3.3). ■

**Remark 3.2.** If we take the supremum over  $\|x\| = \|y\| = 1$  in (3.3) then we get the norm inequality

$$\|f(\alpha V)e\| \|f(\alpha U)e\| \leq \frac{1}{2} \left[ \left\| f(\alpha |V|^2) \right\| \left\| f(\alpha |U|^2) \right\| + \|f(U^*V)\| \right] f(\alpha) \quad (3.6)$$

for any  $e \in H$ ,  $\|e\| = 1$ .

By Schwarz inequality in  $H$  we have

$$\begin{aligned} \|f(\alpha V)e\| \|f(\alpha U)e\| &\geq |\langle f(\alpha V)e, f(\alpha U)e \rangle| \\ &= |\langle (f(\alpha U))^* f(\alpha V)e, e \rangle| \\ &= |\langle f(\alpha U^*) f(\alpha V)e, e \rangle| \end{aligned}$$

giving the inequality

$$|\langle f(\alpha U^*) f(\alpha V)e, e \rangle| \leq \frac{1}{2} \left[ \left\| f(\alpha |V|^2) \right\| \left\| f(\alpha |U|^2) \right\| + \|f(U^*V)\| \right] f(\alpha) \quad (3.7)$$

for any  $e \in H$ ,  $\|e\| = 1$ .

Since  $U$  and  $V$  are normal and  $U^*V = VU^*$ , then  $f(\alpha U^*)$  and  $f(\alpha V)$  are normal and commute implying that  $f(\alpha U^*) f(\alpha V)$  is normal. Taking the supremum over  $\|e\| = 1$  we get

$$\|f(\alpha U^*) f(\alpha V)\| \leq \frac{1}{2} \left[ \left\| f(\alpha |V|^2) \right\| \left\| f(\alpha |U|^2) \right\| + \|f(U^*V)\| \right] f(\alpha). \quad (3.8)$$

**Example 3.3.** a. If  $U, V$  are normal operators on the Hilbert space  $(H; \langle \cdot, \cdot \rangle)$ ,  $U^*V = VU^*$  and  $\alpha > 0$  then

$$\begin{aligned} &|\langle \exp(\alpha V)e, x \rangle \langle \exp(\alpha U)e, y \rangle| \\ &\leq \frac{1}{2} \left[ \left\langle \exp(\alpha |V|^2)x, x \right\rangle \left\langle \exp(\alpha |U|^2)y, y \right\rangle + |\langle \exp(U^*V)x, y \rangle| \right] \exp(\alpha) \end{aligned} \quad (3.9)$$

for any  $x, y, e \in H$  with  $\|e\| = 1$ .

b. If  $U, V$  are normal operators on the Hilbert space  $(H; \langle \cdot, \cdot \rangle)$ ,  $U^*V = VU^*$  and  $\|U\|, \|V\| < 1$ ,  $\alpha \in (0, 1)$  then

$$\begin{aligned} &\left| \left\langle (1_H - \alpha V)^{-1}e, x \right\rangle \left\langle (1_H - \alpha U)^{-1}e, y \right\rangle \right| \\ &\leq \frac{1}{2} \left[ \left\langle (1_H - \alpha |V|^2)^{-1}x, x \right\rangle \left\langle (1_H - \alpha |U|^2)^{-1}y, y \right\rangle \right. \\ &\quad \left. + \left| \left\langle (1_H - U^*V)^{-1}x, y \right\rangle \right| \right] (1 - \alpha)^{-1} \end{aligned} \quad (3.10)$$

for any  $x, y, e \in H$  with  $\|e\| = 1$ .

## References

- [1] J. M. Aldaz, Strengthened Cauchy-Schwarz and Hölder inequalities. *J. Inequal. Pure Appl. Math.* **10** (2009), no. 4, Article 116, 6 pp.
- [2] N. G. de Bruijn, Problem 12, *Wisk. Opgaven*, **21** (1960), 12-14.
- [3] M. L. Buzano, Generalizzazione della diseguaglianza di Cauchy-Schwarz. (Italian), *Rend. Sem. Mat. Univ. e Politech. Torino*, **31** (1971/73), 405-409 (1974).
- [4] A. De Rossi, A strengthened Cauchy-Schwarz inequality for biorthogonal wavelets. *Math. Inequal. Appl.* **2** (1999), no. 2, 263-282.
- [5] S. S. Dragomir, Some refinements of Schwartz inequality, Simpozionul de Matematici și Aplicații, Timișoara, Romania, 1-2 Noiembrie 1985, 13-16.
- [6] S. S. Dragomir, Some inequalities for power series of selfadjoint operators in Hilbert spaces via reverses of the Schwarz inequality. *Integral Transforms Spec. Funct.* **20** (2009), no. 9-10, 757-767.
- [7] S. S. Dragomir, A potpourri of Schwarz related inequalities in inner product spaces. I. *J. Inequal. Pure Appl. Math.* **6** (2005), no. 3, Article 59, 15 pp.
- [8] S. S. Dragomir, A potpourri of Schwarz related inequalities in inner product spaces. II. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 1, Article 14, 11 pp.
- [9] S. S. Dragomir, Reverses of Schwarz, triangle and Bessel inequalities in inner product spaces. *J. Inequal. Pure Appl. Math.* **5** (2004), no. 3, Article 76, 18 pp.
- [10] S. S. Dragomir, New reverses of Schwarz, triangle and Bessel inequalities in inner product spaces. *Aust. J. Math. Anal. Appl.* **1** (2004), no. 1, Art. 1, 18 pp. New York, 2007. xii+243 pp. ISBN: 978-1-59454-903-8; 1-59454-903-6.
- [11] S. S. Dragomir, Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces. *Demonstratio Math.* **40** (2007), no. 2, 411-417.
- [12] S. S. Dragomir, Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces. *Tamkang J. Math.* **39** (2008), no. 1, 1-7.
- [13] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*. Nova Science Publishers, Inc., Hauppauge, NY, 2005. viii+249 pp. ISBN: 1-59454-202-3.
- [14] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces*. Nova Science Publishers, Inc.,
- [15] S. S. Dragomir, *Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces*. Springer Briefs in Mathematics. Springer, 2013. x+120 pp. ISBN: 978-3-319-01447-0; 978-3-319-01448-7.

- [16] S. S. Dragomir and Anca C. Goşa, Quasilinearity of some composite functionals associated to Schwarz's inequality for inner products. *Period. Math. Hungar.* **64** (2012), no. 1, 11–24.
- [17] S. S. Dragomir and B. Mond, Some mappings associated with Cauchy-Buniakowski-Schwarz's inequality in inner product spaces. *Soochow J. Math.* **21** (1995), no. 4, 413–426.
- [18] S. S. Dragomir and I. Sándor, Some inequalities in pre-Hilbertian spaces. *Studia Univ. Babeş-Bolyai Math.* **32** (1987), no. 1, 71–78.
- [19] H. Gunawan, On  $n$ -inner products,  $n$ -norms, and the Cauchy-Schwarz inequality. *Sci. Math. Jpn.* **55** (2002), no. 1, 53–60
- [20] K. E. Gustafson and D. K. M. Rao, *Numerical Range*, Springer-Verlag, New York, Inc., 1997.
- [21] E. R. Lorch, The Cauchy-Schwarz inequality and self-adjoint spaces. *Ann. of Math. (2)* **46**, (1945). 468–473.
- [22] C. Lupu and D. Schwarz, Another look at some new Cauchy-Schwarz type inner product inequalities. *Appl. Math. Comput.* **231** (2014), 463–477.
- [23] M. Marcus, The Cauchy-Schwarz inequality in the exterior algebra. *Quart. J. Math. Oxford Ser. (2)* **17** 1966 61–63.
- [24] P. R. Mercer, A refined Cauchy-Schwarz inequality. *Internat. J. Math. Ed. Sci. Tech.* **38** (2007), no. 6, 839–842.
- [25] F. T. Metcalf, A Bessel-Schwarz inequality for Gramians and related bounds for determinants. *Ann. Mat. Pura Appl. (4)* **68** 1965 201–232.
- [26] T. Precupanu, On a generalization of Cauchy-Buniakowski-Schwarz inequality. *An. Şti. Univ. "Al. I. Cuza" Iaşi Sect. I a Mat. (N.S.)* **22** (1976), no. 2, 173–175.
- [27] K. Trenčevski and R. Malčeski, On a generalized  $n$ -inner product and the corresponding Cauchy-Schwarz inequality. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 2, Article 53, 10 pp.
- [28] G.-B. Wang and J.-P. Ma, Some results on reverses of Cauchy-Schwarz inequality in inner product spaces. *Northeast. Math. J.* **21** (2005), no. 2, 207–211.